MONOTONICITY OF THE POWER FUNCTION OF PILLAI'S TRACE TEST*

by

Michael D. Perlman

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University of Minnesota
Minneapolis, Minnesota

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Abstract

It is shown that the power function of Pillai's trace test for the MANOVA problem is monotonically increasing in each noncentrality parameter provided that the cutoff point is not too large. This result is also true for the problem of testing independence of two sets of variates.

Key words and phrases: MANOVA problem, testing independence of variates, power functions, invariant tests, monotonicity, convexity, Pillai's trace test, $V^{(t)}$.

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1. Pillai's Test for the MANOVA Problem.

Consider the following canonical form of the MANOVA testing problem suitable for studying the power functions of invariant tests. Let $U: p \times r$ and $V: p \times n$ be random matrices. The p-dimensional column vectors $U_1, \ldots, U_r, V_1, \ldots, V_n$ are mutually independent and normally distributed with common nonsingular covariance matrix $\Sigma: p \times p$, and $EU = \Theta: p \times r$, $EV = 0: p \times n$. The problem is to test

$$\Theta = 0 \quad \text{against} \quad \Theta \neq 0. \tag{1.1}$$

This problem is invariant under all transformations of the form

$$(U, V) \to (AU\Gamma_r, AV\Gamma_n) \tag{1.2}$$

where $A: p \times p$ is nonsingular and $\Gamma_r: r \times r$ and $\Gamma_n: n \times n$ are orthogonal. We assume that $p < n + r$ (if $p \geq n + r$, there are no nontrivial invariant tests). A maximal invariant statistic is $(f_1, \ldots, f_t)$, where $t = \min\{p, r\}$ and $1 \geq f_1 \geq \ldots \geq f_t \geq 0$ are the ordered t largest characteristic roots of $UU'(UU' + VV')^{-1}$. A maximal invariant parameter is $(\gamma_1, \ldots, \gamma_t)$, where $1 \geq \gamma_1 \geq \ldots \geq \gamma_t \geq 0$ are the ordered t largest characteristic roots of $\Theta\Theta'\Sigma^{-1}$.

For any region $Q \subseteq R^{p(r+n)}$ define $\pi_Q(\Theta, \Sigma)$ by

$$\pi_Q(\Theta, \Sigma) = P_{\Theta,\Sigma}\{(U, V) \notin Q\},$$

so π_Q is the power function of the test with acceptance region Q. If Q is invariant under all the transformations (1.2) then π_Q is a function of the noncentrality parameters $\gamma_1, \ldots, \gamma_t$, so we write

$$\pi_Q(\Theta, \Sigma) \equiv \varphi_Q(\gamma_1, \ldots, \gamma_t).$$

For any acceptance region $Q \subseteq R^{p(r+n)}$ and for all $i = 1, \ldots, r$ we denote the u_i-section of Q for fixed u_j, $j \neq i$, and fixed v by

$$Q^{(i)}(\tilde{u}_i, v) = \{u_i | (u, v) \in Q\} \subseteq R^p,$$

where $\tilde{u}_i : p \times (r-1) \equiv (u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_r)$.

Das Gupta, Anderson, and Mudholkar [2] have proved the following theorem. They assume that $p \leq n$, but their proof is valid if $p < n + r$.

Theorem 1.1.

Let $Q \subseteq R^{p(r+n)}$ be invariant under all transformations (1.2). Suppose that for each $i = 1, \ldots, r$, $Q^{(i)}(\tilde{u}_i, v)$ is convex, except possibly for $(\tilde{u}_i, v)$ in a set of $p(n+r-1)$-dimensional Lebesgue measure 0. Then $\varphi_Q(\gamma_1, \ldots, \gamma_t)$ increases monotonically in each γ_j.

Consider the following four well-known acceptance regions. The first three are defined only for $p \leq n$, while the last is defined for $p < n + r$.

(i) Roy's maximum root test:

$$Q_1 = \{(u, v) | f_1 \leq k_1\}, \; 0 < k_1 < 1 .$$

(ii) Lawley-Hotelling trace test:

$$Q_2 = \{(u, v) | \sum_1^t f_i (1 - f_i)^{-1} \leq k_2\}, \; 0 < k_2.$$

(iii) Likelihood ratio test:

$$Q_3 = \{(u, v) | \prod_1^t (1 - f_i) \geq k_3\}, \; 0 < k_3 < 1.$$

(iv) Pillai's trace test:

$$Q_4 = \{(u, v) | \sum_1^t f_i \leq k_4\}, \; 0 < k_4 < t.$$

Das Gupta, Anderson, and Mudholkar [2] showed that Q_1, Q_2, and Q_3 satisfy the conditions of Theorem 1.1 and hence their power functions have the monotonicity property. Mudholkar [6] characterized a class of acceptance regions containing Q_1 and Q_2 (but not Q_3 or Q_4) which satisfy the conditions of Theorem 1.1. However, the monotonicity property has not yet been established for the important test Q_4, Pillai's test, based on the statistic $\sum_1^t f_i$, which is usually denoted by $V^{(t)}$. This test is admissible (Schwartz [10]), locally minimax (Schwartz [9]), and proper Bayes provided $n \geq p$ (Kiefer and Schwartz [5]). In this paper we show that $\varphi_{Q_4}(\gamma_1, \ldots, \gamma_t)$ is monotonically increasing in each γ_i provided that the cutoff point k_4 is not too large. The following is our main theorem; its proof is given in Section 3.

Theorem 1.2.

The invariant acceptance region Q_4 satisfies the conditions of Theorem 1.1 if and only if $k_4 \leq \max\{1, p-n\}$. Hence if $k_4 \leq \max\{1, p-n\}$, the power function of Pillai's trace test for the MANOVA problem is monotonically increasing in each noncentrality parameter γ_i.

For $0 < \alpha < 1$ and $p < n + r$, define $k_4(\alpha, p, r, n)$ to be the size α cutoff point for Pillai's test, i.e.,

$$P[\sum_1^t f_i > k_4(\alpha, p, r, n) | \Theta = 0] = \alpha.$$

Then by Theorem 1.2, Theorem 1.1 can be applied to Pillai's test if and only if

$$k_4(\alpha, p, r, n) \leq \max\{1, p-n\}. \tag{1.3}$$

Since $\sum_1^t f_i = \mathrm{tr}[UU'(UU' + VV')^{-1}]$, it is easy to see that $k_4(\alpha, p, r, n)$ is decreasing in α and n, while increasing in r. Furthermore it will be shown in Section 2 that

$$k_4(\alpha, p, r, n) = k_4(\alpha, r, p, n+r-p), \tag{1.4}$$

so that $k_4(\alpha, p, r, n)$ is increasing in p as well.

From this discussion we see that for $n \geq p$ the monotonicity property of the power function of Pillai's test for the MANOVA problem can be deduced from

Theorem 1.1 provided α and n are not too small and p, r are not too large. (Of course, the monotonicity property might hold even if (1.3) is violated, since the sufficient condition of Theorem 1.1 may not be necessary.) Values of $k_4(\alpha, p, r, n)$ have been tabulated by Pillai [7], Pillai and Jayachandran [8], Jayachandran [4], and Fujikoshi [3]. From these sources one can obtain, for example, the smallest value of $n \geq p$ such that $k_4(\alpha, p, r, n) \leq 1$.

2. Pillai's Test For Independence of Variates.

Let $Z: (p_1 + p_2) \times m$ be a random matrix whose columns are independent normal random vectors with mean 0 and common nonsingular covariance matrix Δ. Partition $Z = \binom{X}{Y}$ and $\Delta = \begin{pmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{pmatrix}$ with $X: p_1 \times m$, $Y: p_2 \times m$, $\Delta_{11}: p_1 \times p_1$, etc. In this canonical form, the problem of testing independence of two sets of variates is to test

$$\Delta_{12} = 0 \quad \text{vs.} \quad \Delta_{12} \neq 0. \tag{2.1}$$

The problem is invariant under all transformations of the form

$$\binom{X}{Y} \to \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} \binom{X}{Y} \Gamma, \tag{2.2}$$

where $B_1: p_1 \times p_1$ and $B_2: p_2 \times p_2$ are nonsingular and $\Gamma: m \times m$ is orthogonal. A maximal invariant statistic is $(g_1, \ldots, g_t)$, where $t = \min\{p_1, p_2\}$ and $1 \geq g_1 \geq \ldots \geq g_t \geq 0$ are the ordered t largest characteristic roots of $(XX')^{-1}(XY')(YY')^{-1}(YX') = S_{11}^{-1} S_{12} S_{22}^{-1} S_{21}$, where $S = ZZ'$ is partitioned as Δ. Assume that $m > \max\{p_1, p_2\}$ to insure the nonsingularity of S_{11} and S_{22}, and to insure the existence of nontrivial invariant tests. A maximal invariant parameter is the vector of canonical correlations $(\rho_1, \ldots, \rho_t)$, where $1 \geq \rho_1^2 \geq \ldots \geq \rho_t^2 \geq 0$ are the ordered t largest characteristic roots of $\Delta_{11}^{-1} \Delta_{12} \Delta_{22}^{-1} \Delta_{21}$. The power function of any invariant test is a function of $(\rho_1, \ldots, \rho_t)$.

It is well-known (see Anderson and Das Gupta [1]) that the conditional distribution given S_{22} of the matrix

$$S_{11}^{-1} S_{12} S_{22}^{-1} S_{21} = (S_{11.2} + S_{12} S_{22}^{-1} S_{21})^{-1} S_{12} S_{22}^{-1} S_{21} \tag{2.3}$$

is of the same form as the distribution of the matrix $(UU' + VV')^{-1} UU'$ in the MANOVA problem if we take $(p, r, n) = (p_1, p_2, m - p_2)$, and their (unconditional) null distributions are identical. Using the former fact Anderson and Das Gupta [1] and Mudholkar [6] obtained the following result.

Theorem 2.1.

Suppose that the power of an invariant test for (1.1), which accepts the MANOVA hypothesis $\Theta = 0$ if and only if $(f_1, \ldots, f_t) \in G$, increases monotonically in each noncentrality parameter γ_i. Then the power function of the invariant test for the independence problem (2.1), which accepts the hypothesis $\Delta_{12} = 0$ if and only if $(g_1, \ldots, g_t) \in G$, increases monotonically in each canonical correlation ρ_i.

Combining Theorems 1.1 and 2.1, therefore, it was concluded in [1] and [6] that the power functions of Roy's test based on g_1, the Lawley-Hotelling test based on $\sum_1^t g_i(1-g_i)^{-1}$, and the likelihood ratio test based on $\prod_1^t (1-g_i)$ all have the monotonicity property (these tests are defined only if $m \geq p_1 + p_2$). For all $m > \max(p_1, p_2)$, Theorem 1.2 now provides the following result concerning the power function of Pillai's test for testing independence (2.1). This test accepts $\Delta_{12} = 0$ if and only if

$$\sum_1^t g_i \leq k_4(\alpha, p_1, p_2, m-p_2), \tag{2.4}$$

where k_4 was defined in Section 1.

Theorem 2.2.

Theorems 1.1 and 2.1 together imply the monotonicity property of the power function of Pillai's test for independence if and only if $k_4(\alpha, p_1, p_2, m-p_2) \leq \max\{1, p_1 + p_2 - m\}$.

Since $k_4(\alpha, p_1, p_2, m-p_2)$ decreases with α and m and increases with p_1 and p_2, Pillai's test for independence has the monotonicity property provided α and m are not too small nor p_1 and p_2 too large.

The tables listed in Section 1 can be used to determine the smallest value of $m \geq p_1 + p_2$ such that $k_4(\alpha, p_1, p_2, m-p_2) \leq 1$ for fixed p_1, p_2, and α.

<u>Remark.</u>

Since $S_{21}S_{11}^{-1} S_{12}S_{22}^{-1} = S_{21}S_{11}^{-1} S_{12}(S_{22.1} + S_{21}S_{11}^{-1} S_{12})^{-1}$ and since the t largest characteristic roots of $S_{21}S_{11}^{-1} S_{12}S_{22}^{-1}$ and of $S_{11}^{-1} S_{12}S_{22}^{-1} S_{21}$ are the same, it follows from (2.3) that $k_4(\alpha, p_1, p_2, m-p_2) = k_4(\alpha, p_2, p_1, m-p_1)$, which is equivalent to (1.4).

3. Proof of Theorem 1.2.

Express Q_4 as

$$Q_4 = \{(u, v) | \operatorname{tr}\{uu'(uu' + vv')^{-1}] \leq k_4\}$$

where $0 < k_4 < t$. Since Q_4 is symmetric in the columns $u_1, \ldots, u_r$ of u, it suffices to prove that $Q_4^{(1)}(\tilde{u}_1, v)$ is convex for almost all $(\tilde{u}_1, v)$ if and only if $k_4 \leq \max\{1, p-n\}$. Now

$$\operatorname{tr}[uu'(uu' + vv')^{-1}] = p - \operatorname{tr}[vv'(uu' + vv')^{-1}]$$

and

$$(uu' + vv')^{-1} = (u_1u_1' + \tilde{u}_1\tilde{u}_1' + vv')^{-1} = w^{-1} - \frac{w^{-1}u_1u_1'w^{-1}}{1 + u_1'w^{-1}u_1},$$

where $w = w(\tilde{u}_1, v) = u_1u_1' + vv'$ is nonsingular for almost all $(\tilde{u}_1, v)$ since $p \leq n + r - 1$. Hence, excluding a null set of $(\tilde{u}_1, v)$ values,

$$Q_4^{(1)}(\tilde{u}_1, v) = \{u_1 | \frac{u_1'w^{-1}vv'w^{-1}u_1}{1 + u_1'w^{-1}u_1} \leq \operatorname{tr} vv'w^{-1} - p + k_4\}.$$

Let $\Lambda = \Lambda(\tilde{u}_1, v) = w^{-\frac{1}{2}}vv'w^{-\frac{1}{2}}$, let $y = w^{-\frac{1}{2}}u_1$, and define the region $M = M(\tilde{u}_1, v) \subseteq R^p$ by

$$M(\tilde{u}_1, v) = \{y | \frac{y'\Lambda y}{1 + y'y} \leq \operatorname{tr} \Lambda - p + k_4\}.$$

Thus

$$Q_4^{(1)}(\tilde{u}_1, v) = w^{\frac{1}{2}}[M(\tilde{u}_1, v)]$$

where $w^{\frac{1}{2}}[M] = \{w^{\frac{1}{2}}y | y \in M\}$. Choose $\psi = \psi(\tilde{u}_1, v)$ to be a $p \times p$ orthogonal matrix such that

$$\Lambda = \psi \operatorname{diag}\{\lambda_1, \ldots, \lambda_p\}\psi',$$

where $\lambda_1 \geq \ldots \geq \lambda_p$ are the ordered characteristic roots of Λ $(\lambda_j = \lambda_j(\tilde{u}_1, v))$. Let $z = \psi'y = (z_1, \ldots, z_p)'$ and define the region $H = H(\tilde{u}_1, v) \subseteq R^p$ by

$$H(\tilde{u}_1, v) = \{z | \sum_{j=1}^{p} z_j^2(\lambda_j - \sum_{m=1}^{p}\lambda_m + p - k_4) \leq \sum_{m=1}^{p}\lambda_m - p + k_4\}. \quad (3.1)$$

Then $M(\tilde{u}_1, v) = \psi[H(\tilde{u}_1, v)]$, so that except for a null set of $(\tilde{u}_1, v)$ values,

$$Q_4^{(1)}(\tilde{u}_1, v) = w^{\frac{1}{2}}\psi[H(\tilde{u}_1, v)]. \tag{3.2}$$

Now assume that $k_4 \leq \max\{1, p-n\}$. In view of (3.2) and the linearity of the operator $w^{\frac{1}{2}}\psi$, to verify that Q_4 satisfies the conditions of Theorem 1.1 it suffices to show that $H(\tilde{u}_1, v)$ is convex for all $(\tilde{u}_1, v)$. Since Λ and $I_p - \Lambda$ are positive semidefinite and rank (Λ) = rank $(v) \leq \min\{p, n\} \equiv s$, say, we have that

$$1 \geq \lambda_1 \geq \dots \geq \lambda_s \geq 0 = \lambda_{s+1} = \dots = \lambda_p. \tag{3.3}$$

Hence for each $j = 1,\dots, p$,

$$\begin{aligned} \lambda_j - \sum_{m=1}^{p} \lambda_m + p - k_4 &\geq -\min\{p-1, s\} + p - k_4 \\ &= -\min\{p-1, n\} + p - k_4 \\ &= \max\{1, p-n\} - k_4 \\ &\geq 0. \end{aligned}$$

Therefore from (3.1), $H(\tilde{u}_1, v)$ is an ellipsoid (possibly degenerate or empty) for all $(\tilde{u}_1, v)$ and hence is convex, so Q_4 satisfies the conditions of Theorem 1.1.

Conversely, suppose that $k_4 > \max\{1, p-n\}$. Since $k_4 < t = \min\{p, r\}$, this requires that $r > 1$. Let $\mu = \mu(\tilde{u}_1, v) = \sum_{m=1}^{p} \lambda_m - p + k_4$ and $\beta_j = \beta_j(\tilde{u}_1, v) = \lambda_j - \mu$, so that

$$H(\tilde{u}_1, v) = \{z \mid \sum_{j=1}^{p} z_j^2\beta_j \leq \mu\}.$$

We shall show later that there exists $(\tilde{u}_1, v)$ such that

$$\beta_1(\tilde{u}_1, v) > 0 > \beta_p(\tilde{u}_1, v). \tag{3.4}$$

Since $\beta_j(\tilde{u}_1, v)$ is a continuous function of $(\tilde{u}_1, v)$ there must exist an open set $\Delta \subseteq R^{p(n+r-1)}$ such that (3.4) holds for all $(\tilde{u}_1, v) \in \Delta$. Thus $H(\tilde{u}_1, v)$ fails to be a convex set whenever $(\tilde{u}_1, v) \in \Delta$, which is a non-null set, so Q_4 cannot satisfy the conditions of Theorem 1.1.

We turn to the existence of $(\tilde{u}_1, v)$ satisfying (3.4), which can be rewritten as

$$\sum_{j=2}^{p} \lambda_j < p - k_4 < \sum_{j=1}^{p-1} \lambda_j. \tag{3.5}$$

By assumption, $\max\{1, p-n\} < k_4 < t = \min\{p, r\}$, or equivalently

$$\max\{0, p-r\} < p - k_4 < \min\{p-1, n\}. \tag{3.6}$$

<u>Case (i)</u>: $p \leq n$, $p < r$.

Choose $(\tilde{u}_1, v)$ such that $vv' = I_p$ (the $p \times p$ identity matrix) and $\tilde{u}_1\tilde{u}_1' = \operatorname{diag}\{d_1, \ldots, d_p\}$, where $0 \leq d_1 \leq \ldots \leq d_p$ are defined below. For such $(\tilde{u}_1, v)$ we have

$$(\lambda_1, \ldots, \lambda_p) = ((1 + d_1)^{-1}, \ldots, (1 + d_p)^{-1}),$$

so (3.5) becomes

$$\sum_{j=2}^{p} (1 + d_j)^{-1} < p - k_4 < \sum_{j=1}^{p-1} (1 + d_j)^{-1} . \tag{3.7}$$

Also, (3.6) reduces to $0 < p - k_4 < p - 1$, so $0 \leq \nu \leq p - 2$ where $\nu \equiv [p - k_4]$. If $\nu = 0$ choose the d_j such that $(1 + d_1)^{-1} = 1$ and $(1 + d_j)^{-1} < (p - k_4)/(p-1)$ for $2 \leq j \leq p$. If $1 \leq \nu \leq p - 2$ select θ, δ such that $(p - k_4) - \nu < \theta < 1$ and $0 < \delta < 1 - \theta$, and choose the d_j such that $(1 + d_1)^{-1} = \ldots = (1 + d_\nu)^{-1} = 1$, $(1 + d_{\nu+1})^{-1} = \theta$, and $\sum_{j=\nu+2}^{p} (1 + d_j)^{-1} = \delta$. Then it is easily verified that for all $0 \leq \nu \leq p - 2$, (3.7) is satisfied.

Case (ii): $p \leq n$, $p \geq r$.

Choose $(\tilde{u}_1, v)$ such that $vv' = I_p$ and $\tilde{u}_1\tilde{u}_1' = \text{diag}\{d_1,\dots, d_{r-1},0,\dots, 0\}$ where $0 \leq d_1 \leq \dots \leq d_{r-1}$ are defined below. For such $(\tilde{u}_1, v)$ we have

$$(\lambda_1,\dots, \lambda_p) = (1,\dots, 1, (1+d_1)^{-1},\dots, (1+d_{r-1})^{-1})$$

where there are $p-(r-1)$ ones preceding $(1+d_1)^{-1}$, so (2.5) becomes

$$\sum_{j=1}^{r-1}(1 + d_j)^{-1} < r - k_4 < 1 + \sum_{j=1}^{r-2} (1 + d_j)^{-1}. \tag{3.8}$$

Also, (3.6) reduces to $0 < r - k_4 < r - 1$, so $0 \leq \nu \leq r - 2$ where $\nu \equiv [r - k_4]$. If $\nu = 0$ choose the d_j such that $(1+d_j)^{-1} < (r-k_4)/(r-1)$ for all $j = 1,\dots, r-1$. If $1 \leq \nu \leq r-2$ select θ, δ such that $(r-k_4) - \nu < \theta < 1$ and $0 < \delta < 1-\theta$ and choose the d_j such that $(1+d_1)^{-1} = \dots = (1+d_{\nu-1})^{-1} = 1$, $(1+d_\nu)^{-1} = \theta$, and $\sum_{j=\nu+1}^{r-1} (1+d_j)^{-1} = \delta$. Then (3.8) is satisfied for all $0 \leq \nu \leq r-2$.

Case (iii): $p > n$, $p < r$.

Choose $(\tilde{u}_1, v)$ such that $vv' = \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}$: $p \times p$ and $\tilde{u}_1\tilde{u}_1' = \text{diag}\{d_1,\dots, d_p\}$ where $0 \leq d_1 \leq \dots \leq d_p$ are defined below. For such $(\tilde{u}_1, v)$ we have

$$(\lambda_1,\dots, \lambda_p) = ((1+d_1)^{-1},\dots, (1+d_n)^{-1},0,\dots, 0)$$

so (2.5) becomes

$$\sum_{j=2}^{n} (1+d_j)^{-1} < p - k_4 < \sum_{j=1}^{n} (1+d_j)^{-1} . \tag{3.9}$$

Also, (3.6) reduces to $0 < p - k_4 < n$, so $0 \leq \nu \leq n-1$ where $\nu \equiv [p-k_4]$. If $\nu = 0$ choose the d_j such that $(1+d_1)^{-1} = 1$ and $(1+d_j)^{-1} < (p-k_4)/n$ for $2 \leq j \leq n$. If $1 \leq \nu \leq n-1$ select θ, δ such that $(p-k_4) - \nu < \theta < 1$ and $0 < \delta < 1-\theta$ and choose the d_j such that $(1+d_1)^{-1} = \dots = (1+d_\nu)^{-1}= 1$, $(1+d_{\nu+1})^{-1} = \theta$, and $\sum_{j=\nu+2}^{n} (1+d_j)^{-1} = \delta$. Then (3.9) is satisfied for all $0 \leq \nu \leq n-1$.

Case (iv): $p > n,\ p \geq r$.

Choose $(\tilde{u}_1, v)$ such that $vv' = \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix}: p \times p$ and $\tilde{u}_1\tilde{u}_1' = \mathrm{diag}\{d_1, \ldots, d_{r-1}, 0, \ldots, 0\}$ where $0 \leq d_1 \leq \ldots \leq d_{r-1}$ are defined below. We must consider two subcases.

Suppose first that $n \leq r - 1$. Then for $(\tilde{u}_1, v)$ as defined above we have

$$(\lambda_1, \ldots, \lambda_p) = ((1+d_1)^{-1}, \ldots, (1+d_n)^{-1}, 0, \ldots, 0)$$

so (3.5) reduces to (3.9) and we choose the d_j as in Case (iii).

Suppose next that $n > r - 1$. Then we have that

$$(\lambda_1, \ldots, \lambda_p) = (1, \ldots, 1, (1+d_1)^{-1}, \ldots, (1+d_{r-1})^{-1}, 0, \ldots, 0)$$

where there are $n - (r-1)$ ones preceding $(1+d_1)^{-1}$, so (3.5) becomes

$$\sum_{j=1}^{r-1} (1+d_j)^{-1} < p + r - n - k_4 < 1 + \sum_{j=1}^{r-1} (1+d_j)^{-1}. \tag{3.10}$$

Note that (3.6) reduces to $p - n < p + r - n - k_4 < r$ in this case, so $1 \leq p-n \leq \nu \leq r-1$ where $\nu = [p+r-n-k_4]$. Select θ, δ such that $(p+r-n-k_4) - \nu < \theta < 1$ and $0 < \delta < 1 - \varepsilon$. If we choose the d_j such that $(1+d_1)^{-1} = \ldots = (1+d_{\nu-1})^{-1} = 1$, $(1+d_\nu)^{-1} = \theta$, and $\sum_{j=\nu+1}^{r-1} (1+d_j)^{-1} = \delta$, then (3.10) is satisfied. This completes the proof.

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